

Hardy inequalities in globally twisted waveguides

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Abstract

We establish various Hardy-type inequalities for the Dirichlet Laplacian in perturbed periodically twisted tubes of non-circular cross-sections. We also state conjectures about the existence of such inequalities in more general regimes, which we support by heuristic and numerical arguments.

1 Introduction

“*I have never done anything ‘useful’. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.*” This is a quotation from a 1940 essay [12] by British mathematician G. H. Hardy. Despite of this self-identification as a pure mathematician, his work has found important applications in physics, including the celebrated *Hardy inequality* [11]

$$\forall \varphi \in H_0^1((0, \infty)), \quad \int_0^\infty |\varphi'(x)|^2 dx \geq \frac{1}{4} \int_0^\infty \frac{|\varphi(x)|^2}{|x|^2} dx. \quad (1)$$

For instance, using (1) in the radial component of the three-dimensional Laplacian, the inequality directly explains the *stability* of hydrogen-type atoms in quantum mechanics.

In a different quantum-mechanical context, Ekholm, Kovařík and the last author employed (1) to prove in [7] certain stability of the spectrum of the Dirichlet Laplacian in *locally twisted tubes*

$$\Omega := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta(x_1) & \sin \theta(x_1) \\ 0 & -\sin \theta(x_1) & \cos \theta(x_1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid (x_1, x_2, x_3) \in \mathbb{R} \times \omega \right\}. \quad (2)$$

Here the cross-section ω is an arbitrary bounded open connected set in \mathbb{R}^2 and $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is the twisting angle. Assuming that $\dot{\theta}$ is a compactly supported continuous function with bounded derivative, the authors of [7] derived a waveguide-type analogue of (1), namely,

$$\forall \psi \in H_0^1(\Omega), \quad \int_{\Omega} |\nabla \psi(x)|^2 dx - E_1 \int_{\Omega} |\psi(x)|^2 dx \geq c_H \int_{\Omega} \frac{|\psi(x)|^2}{1 + |x|^2} dx. \quad (3)$$

Here E_1 denotes the first eigenvalue of the Dirichlet Laplacian in ω , $-\Delta_D^\omega$. The constant c_H is positive if, and only if, $\dot{\theta}$ is not identically zero and ω is not rotationally invariant with respect to the origin in \mathbb{R}^2 . If Ω is twisted *locally* in the sense that $\dot{\theta}$ vanishes at infinity, then the spectrum of the Dirichlet Laplacian in Ω , $-\Delta_D^\Omega$, equals $[E_1, \infty)$. Consequently, (3) implies that this spectrum is stable against small short-range perturbations of the Laplacian whenever the tube is locally twisted so that $c_H > 0$. This is the spectral stability, which has applications to quantum transport in waveguide-shaped nanostructures.

Various generalisations of the Hardy inequality (3) has been established in [17, 20, 18]. In addition to the quantum-waveguide context, inequality (3) has been also applied to the study of the large-time behaviour of the heat equation in twisted tubes in [20, 10]. Other effects of twisting has been studied in [9, 16, 15, 9, 4, 21, 14, 3].

In this paper, we are interested in the existence of Hardy inequalities in situations when the tube (2) exhibits a twist which is not necessarily local, *i.e.* $\dot{\theta}$ may not vanish at infinity. Indeed, throughout this paper, we assume that

$$\dot{\theta}(x_1) = \beta + \varepsilon(x_1), \quad (4)$$

where β is a real constant and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ is a (not necessarily small) bounded function (typically vanishing at infinity).

If $\varepsilon = 0$, then Ω is periodically twisted and the spectral problem can be solved by a Floquet-type decomposition. It is shown in [9, 4] that, in this case, $\sigma(-\Delta_D^\Omega) = [\lambda_1, \infty)$, where λ_1 is the lowest eigenvalue of $-\Delta_D^\omega - \beta^2 \partial_\tau^2$ in $L^2(\omega)$, with $\partial_\tau := x_3 \partial_2 - x_2 \partial_3$ being the transverse angular derivative. We have the variational characterisation

$$\lambda_1 = \inf_{\chi \in C_0^\infty(\omega) \setminus \{0\}} \frac{\|\nabla' \chi\|_{L^2(\omega)}^2 + \beta^2 \|\partial_\tau \chi\|_{L^2(\omega)}^2}{\|\chi\|_{L^2(\omega)}^2}, \quad (5)$$

where $\nabla' := (\partial_2, \partial_3)$ stands for the transverse gradient. Here and in the sequel, we keep the coordinate notation introduced in (2), writing $x' := (x_2, x_3) \in \omega$ for the “transverse” coordinates, while $x_1 \in \mathbb{R}$ stands for the “longitudinal” coordinate.

For $\varepsilon \neq 0$ but vanishing at infinity, we always have (*cf.* [4, Sec. 4.1])

$$\sigma_{\text{ess}}(-\Delta_D^\Omega) = [\lambda_1, \infty), \quad (6)$$

however, there might be also discrete eigenvalues below λ_1 . Indeed, it is shown in [9] that the discrete spectrum is not empty provided that the twist is locally “slowed down”, *i.e.*,

$$\int_{\mathbb{R}} (\dot{\theta}^2(x_1) - \beta^2) dx_1 < 0. \quad (7)$$

Recalling (4), this condition is for instance true if $\beta\varepsilon$ is non-positive and not identically equal to zero and ε is small in the supremum norm with respect to β . Further properties of the discrete spectrum are studied in [4].

Our objective is to show that there are Hardy-type inequalities

$$-\Delta_D^\Omega - \lambda_1 \geq \rho(\cdot), \quad (8)$$

with a non-trivial function $\rho : \mathbb{R} \rightarrow [0, \infty)$ in opposite regimes to (7). In particular, there is no discrete spectrum. Note that (3) is a version of (8) if $\beta = 0$, since $\lambda_1 = E_1$ in this case. More precisely, we make the following conjectures.

Conjecture 1. (8) holds if $\beta\varepsilon$ is non-negative and ε is not identically equal to zero.

Conjecture 2. (8) holds if we replace $\varepsilon \mapsto \alpha\varepsilon$, ε is not identically equal to zero and the coupling parameter α is sufficiently large in absolute value.

We say that the twist is *repulsive* if β and ε are such as supposed in Conjecture 1. Note that we impose no sign restrictions in Conjecture 2.

Unfortunately, we have not been able to prove the conjectures in the full generality. In this paper, we establish Conjecture 1 under the additional assumption that the twist is small in a suitable sense. Among the variety of Hardy inequalities proved below, we point out the following result here.

Theorem 1. Let θ be given by (4), where ε is a bounded function. Assume $\beta\varepsilon \geq 0$ and $\beta\varepsilon \neq 0$. Suppose that ω is not rotationally invariant and that its boundary $\partial\omega$ is of class C^4 . There exist positive constants $\beta^* = \beta^*(\omega)$, $\varepsilon^* = \varepsilon^*(\beta, \varepsilon, \omega)$ and $c = c(\beta, \varepsilon, \omega)$ such that if $|\beta| \leq \beta^*$ and $\|\varepsilon\|_\infty \leq \varepsilon^*$ then

$$\forall \psi \in H_0^1(\Omega), \quad \int_\Omega |\nabla \psi(x)|^2 dx - \lambda_1 \int_\Omega |\psi(x)|^2 dx \geq c \int_\Omega \frac{|\psi(x)|^2}{1 + |x|^2} dx. \quad (9)$$

The validity of Conjecture 2 is only supported by heuristic arguments and numerical experiments presented in the following section.

The organisation of the paper is as follows. Section 2 is devoted to mostly non-rigorous arguments supporting the validity of Conjectures 1 and 2. Various Hardy inequalities related to Conjecture 1, in particular that of Theorem 1, are derived in a long Section 3 divided into many subsections. In Appendix A we give a proof of positivity for a one-dimensional Schrödinger operator (*cf.* Proposition 1) which we use as a support for the validity of Conjecture 2 in Section 2. Finally, in Appendix B we explain why the tool of Neumann bracketing is not suitable for the proof of Hardy inequalities in the present setting.

2 Heuristic arguments and numerics

The validity of Conjectures 1 and 2 is supported by the following arguments.

2.1 Thin-width asymptotics

Given a positive number δ , let us denote by Ω_δ the tube (2) where ω is replaced by the scaled domain $\delta\omega := \{\delta x' \mid x' \in \omega\}$. The behaviour of the spectrum of

$-\Delta_D^{\Omega_\delta}$ as $\delta \rightarrow 0$ has been investigated in [2, 6, 19]. In the last reference it is proved that the limit

$$-\Delta_D^{\Omega_\delta} - \delta^{-2}E_1 \xrightarrow{\delta \rightarrow 0} -\Delta^{\mathbb{R}} + C_\omega \dot{\theta}^2 \quad (10)$$

holds in a norm resolvent sense after a suitable identification of Hilbert spaces. Here $-\Delta^{\mathbb{R}}$ denotes the one-dimensional Laplacian in $L^2(\mathbb{R})$ with $\mathfrak{D}(-\Delta^{\mathbb{R}}) := H^2(\mathbb{R})$ and $C_\omega := \|\partial_\tau \mathcal{J}_1\|_{L^2(\omega)}^2$, where \mathcal{J}_1 is a normalised eigenfunction of $-\Delta_D^\omega$ corresponding to E_1 . Note that C_ω is positive if, and only if, ω is not rotationally invariant with respect to the origin in \mathbb{R}^2 .

Let $\lambda_1(\delta)$ denote the eigenvalue (5) where ω is replaced by $\delta\omega$. Using the asymptotics $\lambda_1(\delta) = \delta^{-2}E_1 + C_\omega\beta^2 + \mathcal{O}(\delta^2)$ as $\delta \rightarrow 0$, (10) yields

$$-\Delta_D^{\Omega_\delta} - \lambda_1(\delta) \xrightarrow{\delta \rightarrow 0} -\Delta^{\mathbb{R}} + C_\omega (\dot{\theta}^2 - \beta^2) \quad (11)$$

in the norm resolvent sense. It follows that (7) is indeed sufficient for the existence of discrete eigenvalues, at least in the regime of small δ . On the other hand, no discrete spectrum is expected if the expression $\dot{\theta}^2 - \beta^2 = \varepsilon^2 + 2\beta\varepsilon$ is non-negative. This is obviously the case if $\beta\varepsilon$ is non-negative (Conjecture 1).

At the same time, replacing $\varepsilon \mapsto \alpha\varepsilon$ in (4) and considering the resulting θ in the potential on the right hand side (11), we end up with the one-dimensional Schrödinger operator

$$H_\alpha := -\Delta^{\mathbb{R}} + C_\omega (\alpha^2\varepsilon^2 + 2\alpha\beta\varepsilon) \quad (12)$$

depending on the coupling constant $\alpha \in \mathbb{R}$. Under some hypotheses about ε , H_α has no negative spectrum if the coupling $|\alpha|$ is large enough (Conjecture 2). This is obvious for ε of the shape of a characteristic function. In general, the problem is to take into account the small intervals around points where (continuous) ε vanishes. In Appendix A we prove the following sufficient condition.

Proposition 1. *Let ε be a continuous function whose support is a closure of a finite union of bounded open intervals. Then there exists a non-negative number α_0 depending on β , C_ω and properties of ε such that $H_\alpha \geq 0$ for all $|\alpha| \geq \alpha_0$.*

The question is to extend these asymptotic results to non-infinitesimally small δ and to cast the operator positivity to the existence of Hardy inequalities.

2.2 Geometric considerations

The following heuristic argument relies on one's imagination only. Assume that the function ε has a piecewise constant profile: $\varepsilon(x_1) = \alpha > 0$ if $|x_1| < x_1^0$, with some positive x_1^0 , and $\varepsilon(x_1) = 0$ elsewhere.

If the origin of \mathbb{R}^2 lies outside ω , Ω converges in a geometric sense as $|\alpha| \rightarrow \infty$ to the disjoint union of two semi-tubes $\Omega_\pm := \Omega \cap \{\pm x_1 > x_1^0\}$. By adapting the proof of Section 3.3, it is easy to see that the spectrum of the Dirichlet Laplacian in any of the semi-tubes does not start below λ_1 . Moreover, because of the presence of the extra Dirichlet conditions at $\{|x_1| = x_1^0\}$ and (1), the shifted operators $-\Delta_D^{\Omega_\pm} - \lambda_1$ will satisfy a Hardy inequality of the type (8).

If the origin of \mathbb{R}^2 lies inside ω , Ω converges as $\alpha \rightarrow \infty$ to a set composed of Ω_- , Ω_+ and a connecting tubular channel of radius $\text{dist}(0, \partial\omega)$. Again, it

should be possible to show that the Dirichlet Laplacian shifted by λ_1 satisfies a Hardy inequality in this domain.

The above arguments give a strong geometric support for the validity of Conjecture 2, at least in the case of the special profile of ε .

2.3 Numerical simulations

Finally, we have performed several numerical experiments to support the validity of Conjectures 1 and 2. For illustration, let us take a square cross-section $\omega = (-\frac{1}{2}, \frac{1}{2})^2$ and $\beta = 1$. In front of ε in (4), we add the coupling constant $\alpha \in \mathbb{R}$ and consider two kinds of profile of ε :

$$\varepsilon_1(x_1) := \begin{cases} 1 & \text{if } |x_1| \leq 1, \\ 0 & \text{elsewhere,} \end{cases} \quad \varepsilon_2(x_1) := \begin{cases} 1 - |x_1| & \text{if } |x_1| \leq 1, \\ 0 & \text{elsewhere.} \end{cases} \quad (13)$$

Our numerical calculations have been done in a finite tube $\Omega_L := \Omega \cap \{|x_1| < L\}$ with L so large that the computed eigenvalues stop to be sensitive to further enlargements of L .

In Figure 1 we present the dependence of the lowest eigenvalue of $-\Delta_D^{\Omega_L}$ on α as the blue curve. The horizontal red line corresponds to the energy λ_1 , which is the threshold of the essential spectrum of $-\Delta_D^{\Omega}$, cf. (6). Hence, the blue curve below the red horizontal line approximates the lowest discrete eigenvalue of $-\Delta_D^{\Omega}$, while there is just the essential spectrum of $-\Delta_D^{\Omega}$ above the red line. Consequently, the validity of a Hardy-type inequality (8) is expected whenever the blue curve is strictly above the red line.

We see that the numerical pictures clearly confirm our Conjectures 1 and 2. Indeed, $\inf \sigma(-\Delta_D^{\Omega_L}) > \lambda_1$ whenever $\alpha > 0$ (Conjecture 1) or $\alpha < \alpha^* < 0$ (Conjecture 2). We approximately have $\alpha_1^* \approx -2$ and $\alpha_2^* \approx -3$ for the profile ε_1 and ε_2 , respectively. It is remarkable that the critical values of α^* are so close to the smallest values of α for which the sufficient condition (7) applies. Indeed, (7) yields the existence of a discrete spectrum of $-\Delta_D^{\Omega}$ whenever $\alpha \in (-2, 0)$ and $\alpha \in (-3, 0)$ for the profile ε_1 and ε_2 , respectively.

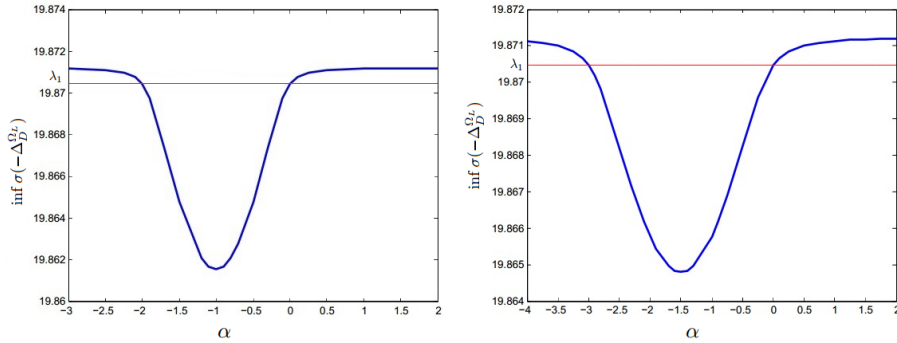


Figure 1: Dependence of $\inf \sigma(-\Delta_D^{\Omega_L})$ on α for the two profiles (13) and $L = 100$. The horizontal (red) line corresponds to the energy $\lambda_1 = \inf \sigma_{\text{ess}}(-\Delta_D^{\Omega})$.

3 Hardy inequalities for a repulsive twist

This long section divided into many subsections is primarily intended to establish a well-arranged proof of Theorem 1, which deals with $\beta\varepsilon \geq 0$. However, some of the intermediate results might be interesting on its own and without this sign (and other) restriction(s).

3.1 Curvilinear coordinates

The very definition (2) gives rise to a diffeomorphism between Ω and the straight tube $\mathbb{R} \times \omega$. Passing to the curvilinear coordinates $(x_1, x_2, x_3) \in \mathbb{R} \times \omega$, the Dirichlet Laplacian $-\Delta_D^\Omega$ in $L^2(\Omega)$ can be identified (*cf.* [17] for more details) with the operator H in $L^2(\mathbb{R} \times \omega)$ associated with the quadratic form

$$h[\psi] := \|\partial_1 \psi - \dot{\theta} \partial_\tau \psi\|^2 + \|\nabla' \psi\|^2, \quad \mathfrak{D}(h) := H_0^1(\mathbb{R} \times \omega). \quad (14)$$

Here and in the sequel $\|\cdot\|$ denotes the norm of $L^2(\mathbb{R} \times \omega)$. The associated inner product will be denoted by (\cdot, \cdot) .

Since $\dot{\theta}$ is bounded, the space $C_0^\infty(\mathbb{R} \times \omega)$ is a core of h . Henceforth we thus take an arbitrary

$$\psi \in C_0^\infty(\mathbb{R} \times \omega).$$

Moreover, since H commutes with complex conjugation, we may suppose that ψ is real-valued.

3.2 Ground-state decomposition

Let χ denote an eigenfunction of $-\Delta_D^\omega - \beta^2 \partial_\tau^2$ corresponding to λ_1 . We choose χ positive and normalised to 1 in $L^2(\omega)$. Since we are interested in properties of H near the threshold λ_1 of the essential spectrum, it is useful to make the decomposition

$$\psi(x) = \chi(x') \phi(x), \quad (15)$$

where ϕ is a $C_0^\infty(\mathbb{R} \times \omega)$ function actually defined by (15).

It is straightforward to check that

$$\begin{aligned} Q[\psi] &:= h[\psi] - \lambda_1 \|\psi\|^2 \\ &= \|\partial_1 \psi - \varepsilon \partial_\tau \psi - \beta \chi \partial_\tau \phi\|^2 + \|\chi \nabla' \phi\|^2 - 2(\partial_1 \psi - \varepsilon \partial_\tau \psi, \beta \phi \partial_\tau \chi) \\ &= \|\partial_1 \psi\|^2 + \|\varepsilon \partial_\tau \psi\|^2 + \|\chi \nabla' \phi\|^2 + \|\beta \chi \partial_\tau \phi\|^2 \\ &\quad + 2(\partial_\tau \psi, \beta \varepsilon \partial_\tau \psi) - 2(\partial_1 \psi, \varepsilon \partial_\tau \psi) - 2(\partial_1 \psi, \beta \partial_\tau \psi). \end{aligned} \quad (16)$$

Here and in the sequel, we use the same symbol ε (respectively χ) for the function $\varepsilon \otimes 1$ (respectively $1 \otimes \chi$) on $\mathbb{R} \times \omega$, and similarly for other functions that will appear below.

3.3 Positivity for a trivial twist

It is not clear from (16) whether $Q[\psi] \geq 0$ if $\beta\varepsilon \geq 0$. In fact, the non-negativity is not completely obvious even for the trivial situation $\varepsilon = 0$ (periodically twisted tube), but it can be established as follows.

If $\varepsilon = 0$, then (16) reduces to

$$Q[\psi] = \|\partial_1 \psi\|^2 + \|\chi \nabla' \phi\|^2 + \|\beta \chi \partial_\tau \phi\|^2 - 2(\partial_1 \psi, \beta \partial_\tau \psi). \quad (17)$$

We write the mixed term in (17) as follows

$$-2(\partial_1 \psi, \beta \partial_\tau \psi) = -2(\partial_1 \psi, \beta \chi \partial_\tau \phi) - 2(\partial_1 \psi, \beta \phi \partial_\tau \chi). \quad (18)$$

By an integration by parts in the first variable, the second term on the right hand side is equal to zero:

$$-2(\partial_1 \psi, \beta \phi \partial_\tau \chi) = -\frac{1}{2} \beta \int_{\mathbb{R} \times \omega} (\partial_1 \phi^2) (\partial_\tau \chi^2) = 0. \quad (19)$$

Consequently,

$$Q[\psi] = \|\partial_1 \psi - \beta \chi \partial_\tau \phi\|^2 + \|\chi \nabla' \phi\|^2 \geq 0. \quad (20)$$

That is, $H - \lambda_1 \geq 0$.

Except for $\psi = 0$ (*i.e.* $\phi = 0$), inequality (20) is always strict. On the other hand, (20) becomes sharp asymptotically as $N \rightarrow \infty$ when considering the sequence of functions $\psi_N(x) = \phi_N(x_1) \chi(x')$, where $\phi_N(x_1) := 1$ if $|x_1| \leq N$, $\phi_N(x_1) := (2N - |x_1|)/N$ if $N < |x_1| < 2N$, and $\phi_N = 0$ otherwise. Consequently, $H - \lambda_1$ is critical in the sense that adding to $H - \lambda_1$ an arbitrarily small non-positive smooth potential which is not identically equal to zero leads to the appearance of negative spectrum. This shows that we cannot have a Hardy inequality (8) for $\varepsilon = 0$.

3.4 Positivity for a small repulsive twist

Using (19), we rewrite the second line of (16) as follows

$$Q[\psi] = \|\partial_1 \psi - \varepsilon \partial_\tau \psi - \beta \chi \partial_\tau \phi\|^2 + \|\chi \nabla' \phi\|^2 + 2(\varepsilon \phi \partial_\tau \chi, \beta \phi \partial_\tau \chi) + 2(\varepsilon \chi \partial_\tau \phi, \beta \phi \partial_\tau \chi). \quad (21)$$

Note that the last but one integral on the right hand side is non-negative whenever $\beta \varepsilon \geq 0$, in particular for any repulsive twist. We assume this sign restriction henceforth. The last integral (of indefinite sign) can be estimated by means of the Schwarz and Young inequalities

$$\begin{aligned} 2(\varepsilon \chi \partial_\tau \phi, \beta \phi \partial_\tau \chi) &\geq -2 \|\sqrt{\beta \varepsilon} \phi \partial_\tau \chi\| \|\sqrt{\beta \varepsilon} \chi \partial_\tau \phi\| \\ &\geq -\delta \|\sqrt{\beta \varepsilon} \phi \partial_\tau \chi\|^2 - \frac{1}{\delta} \|\sqrt{\beta \varepsilon} \chi \partial_\tau \phi\|^2, \end{aligned} \quad (22)$$

with any positive δ . Here the first term on the right hand side can be controlled by the last but one integral on the right hand side of (21). The second term on the right hand side of (22) can be estimated using the pointwise estimate

$$|\partial_\tau \phi| \leq a |\nabla' \phi|, \quad a := \sup_{x' \in \omega} |x'|, \quad (23)$$

and controlled by the second term on the right hand side of (21) provided that $\beta \varepsilon$ is small. More specifically, we thus have

$$Q[\psi] \geq \|\partial_1 \psi - \varepsilon \partial_\tau \psi - \beta \chi \partial_\tau \phi\|^2 + (2 - \delta) \|\sqrt{\beta \varepsilon} \phi \partial_\tau \chi\|^2 + \left[1 - \frac{\|\beta \varepsilon\|_\infty a^2}{\delta}\right] \|\chi \nabla' \phi\|^2. \quad (24)$$

Consequently, choosing $\delta = 2$, we conclude with the desired positivity:

Proposition 2. $H - \lambda_1 \geq 0$ provided that $\beta\varepsilon \geq 0$ and

$$\|\beta\varepsilon\|_\infty a^2 \leq 2. \quad (25)$$

3.5 Local Hardy inequality

In addition to $\beta\varepsilon \geq 0$, let us now assume that $\beta\varepsilon$ is non-trivial, so that we are in the situation of repulsive global twist (Hardy inequalities for the local twist, *i.e.* $\beta = 0$, such as (3) are known, see the introduction). We also strengthen (25) to

$$\|\beta\varepsilon\|_\infty a^2 < 2. \quad (26)$$

Choosing in (24) $\delta = \|\beta\varepsilon\|_\infty a^2$, neglecting the first term on the right hand side and recalling the decomposition (15), we get

$$Q[\psi] \geq (2 - \|\beta\varepsilon\|_\infty a^2) \left\| \sqrt{\beta\varepsilon} \frac{\partial_\tau \chi}{\chi} \psi \right\|^2. \quad (27)$$

This inequality has been established for any $\psi \in C_0^\infty(\mathbb{R} \times \omega)$, however, by density it extends to all $\psi \in H_0^1(\mathbb{R} \times \omega)$. It thus represents a Hardy-type inequality (8) that we state in the following theorem.

Theorem 2. *Let $\beta\varepsilon \geq 0$ and $\beta\varepsilon \neq 0$. Then*

$$H - \lambda_1 \geq (2 - \|\beta\varepsilon\|_\infty a^2) \beta\varepsilon \left(\frac{\partial_\tau \chi}{\chi} \right)^2. \quad (28)$$

It is a *local* Hardy inequality (*cf.* [17] for the terminology) if (26) holds, since ε can be compactly supported in \mathbb{R} . Moreover, $\partial_\tau \chi$ can vanish in $\bar{\omega}$. However, it is important to notice that $\partial_\tau \chi$ cannot vanish on a subset of ω with positive measure, unless ω is rotationally invariant. Indeed, by differentiating the equation for χ and noticing that ∂_τ commutes with Δ' , the function $\eta := \partial_\tau \chi$ satisfies the same equation $(-\Delta' - \beta^2 \partial_\tau^2) \eta = \lambda_1 \eta$ in ω for which the unique continuation property holds. Finally, let us notice that the function on the right hand side of (28) can diverge on $\partial\Omega$ due to the presence of χ in the denominator, which makes a resemblance to the classical Hardy inequality (1).

3.6 A very brute estimate

We continue assuming $\beta\varepsilon \geq 0$ and $\beta\varepsilon \neq 0$. Our objective is to cast (28) into a *global* Hardy inequality, *i.e.* with a right hand side being a positive function in $\mathbb{R} \times \omega$. This can be done by employing the presence of $\|\partial_1 \psi\|$ and $\|\chi \nabla' \phi\|$ in (24).

We thus come back to the second equality in (16), employ (19) and further develop the expression as follows

$$\begin{aligned} Q[\psi] &= \|\partial_1 \psi\|^2 + \|\varepsilon \chi \partial_\tau \phi\|^2 + \|\varepsilon \partial_\tau \chi \phi\|^2 + \|\chi \nabla' \phi\|^2 + \|\beta \chi \partial_\tau \phi\|^2 \\ &\quad + 2\|\sqrt{\beta\varepsilon} \chi \partial_\tau \phi\|^2 + 2\|\sqrt{\beta\varepsilon} \partial_\tau \chi \phi\|^2 + 4(\sqrt{\beta\varepsilon} \chi \partial_\tau \phi, \sqrt{\beta\varepsilon} \partial_\tau \chi \phi) \\ &\quad + 2(\varepsilon \chi \partial_\tau \phi, \varepsilon \partial_\tau \chi \phi) - 2(\partial_1 \psi, \varepsilon \partial_\tau \chi \phi) - 2(\partial_1 \psi, (\varepsilon + \beta) \chi \partial_\tau \phi). \end{aligned} \quad (29)$$

Using the brute estimates

$$\begin{aligned}
|2(\partial_1 \psi, \varepsilon \partial_\tau \chi \phi)| &\leq \delta_1 \|\partial_1 \psi\|^2 + \delta_1^{-1} \|\varepsilon \partial_\tau \chi \phi\|^2, \\
|2(\partial_1 \psi, (\varepsilon + \beta) \chi \partial_\tau \phi)| &\leq \delta_2 \|\partial_1 \psi\|^2 + \delta_2^{-1} \|(\varepsilon + \beta) \chi \partial_\tau \phi\|^2, \\
|2(\sqrt{\beta \varepsilon} \chi \partial_\tau \phi, \sqrt{\beta \varepsilon} \partial_\tau \chi \phi)| &\leq \delta_3 \|\sqrt{\beta \varepsilon} \partial_\tau \chi \phi\|^2 + \delta_3^{-1} \|\sqrt{\beta \varepsilon} \chi \partial_\tau \phi\|^2, \\
|2(\varepsilon \chi \partial_\tau \phi, \varepsilon \partial_\tau \chi \phi)| &\leq \delta_4 \|\varepsilon \partial_\tau \chi \phi\|^2 + \delta_4^{-1} \|\varepsilon \chi \partial_\tau \phi\|^2,
\end{aligned}$$

with arbitrary positive numbers $\delta_1, \delta_2, \delta_3, \delta_4$, we obtain

$$\begin{aligned}
Q[\psi] &\geq (1 - \delta_1 - \delta_2) \|\partial_1 \psi\|^2 + \|\chi \nabla' \phi\|^2 \\
&\quad + \int_{\mathbb{R} \times \omega} [(1 - \delta_1^{-1} - \delta_4) \varepsilon^2 + 2(1 - \delta_3) \beta \varepsilon] |\phi \partial_\tau \chi|^2 dx \\
&\quad + \int_{\mathbb{R} \times \omega} [(1 - \delta_2^{-1}) \beta^2 + (1 - \delta_2^{-1} - \delta_4^{-1}) \varepsilon^2 + 2(1 - \delta_2^{-1} - \delta_3^{-1}) \beta \varepsilon] |\chi \partial_\tau \phi|^2 dx.
\end{aligned}$$

Choosing $\delta_1 = \delta_2 = 1/4$, $\delta_3 = 1/2$ and $\delta_4 = 1$, the previous inequality reads

$$\begin{aligned}
Q[\psi] &\geq \frac{1}{2} \|\partial_1 \psi\|^2 + \|\chi \nabla' \phi\|^2 + \int_{\mathbb{R} \times \omega} (\beta \varepsilon - 4\varepsilon^2) |\phi \partial_\tau \chi|^2 dx \\
&\quad - \int_{\mathbb{R} \times \omega} (3\beta^2 + 4\varepsilon^2 + 10\beta \varepsilon) |\chi \partial_\tau \phi|^2 dx.
\end{aligned}$$

Finally, employing the pointwise bound (23), we conclude with

$$Q[\psi] \geq \frac{1}{2} \|\partial_1 \psi\|^2 + c_1 \|\chi \nabla' \phi\|^2 + c_2 \|\sqrt{\beta \varepsilon} \partial_\tau \chi \phi\|^2, \quad (30)$$

where

$$c_1 = 1 - a^2 (3\beta^2 + 4\|\varepsilon\|_\infty^2 + 10\|\beta \varepsilon\|_\infty), \quad c_2 = 1 - 4 \frac{\|\varepsilon\|_\infty}{|\beta|}, \quad (31)$$

are positive constants provided that

$$4\|\varepsilon\|_\infty < |\beta| < \frac{2}{a\sqrt{23}}. \quad (32)$$

This is a condition on the smallness of both the global periodic twist β and its local perturbation ε .

3.7 An auxiliary transverse problem

For any number $\epsilon \in \mathbb{R}$, define

$$\mu_\epsilon := \inf_{\phi \in C_0^\infty(\omega) \setminus \{0\}} \frac{\|\chi \nabla' \phi\|_{L^2(\omega)}^2 + \epsilon^2 \|\partial_\tau \chi \phi\|_{L^2(\omega)}^2}{\|\chi \phi\|_{L^2(\omega)}^2}. \quad (33)$$

Consider the quadratic form

$$q_\epsilon[\phi] := \|\chi \nabla \phi\|_{L^2(\omega)}^2 + \epsilon^2 \|\partial_\tau \chi \phi\|_{L^2(\omega)}^2, \quad \mathfrak{D}(q_\epsilon) := C_0^\infty(\omega),$$

in the Hilbert space $L^2(\omega, \chi(x')^2 dx')$ and denote by \tilde{q}_ϵ its closure. Then μ_ϵ is the lowest point in the spectrum of the self-adjoint operator h_ϵ in $L^2(\omega, \chi(x')^2 dx')$ associated with \tilde{q}_ϵ . Our objective is to show that μ_ϵ is positive unless $\epsilon = 0$ or ω is rotationally symmetric with respect to the origin.

Lemma 1. *Let $\partial\omega$ be of class C^4 . There exists a positive constant ϵ_0 , depending on the geometry of ω , such that for all $|\epsilon| < \epsilon_0$, h_ϵ is an operator with compact resolvent.*

Proof. Let us introduce the unitary transform

$$U : L^2(\omega, \chi(x')^2 dx') \rightarrow L^2(\omega) : \{\phi \mapsto \chi\phi\},$$

which is well defined because χ is positive in ω . Then h_ϵ is unitarily equivalent to the operator $\hat{h}_\epsilon := Uh_\epsilon U^{-1}$ in $L^2(\omega)$. The latter is the operator associated in $L^2(\omega)$ with the quadratic form

$$\hat{q}_\epsilon[\psi] := \tilde{q}_\epsilon[U^{-1}\psi], \quad \mathfrak{D}(\hat{q}_\epsilon) := U\mathfrak{D}(\tilde{q}_\epsilon).$$

Notice that the space $C_0^\infty(\omega)$, which is a core of \tilde{q}_ϵ , is left invariant by both U and U^{-1} . For any $\psi \in C_0^\infty(\omega)$, by integrating by parts, it is easy to verify

$$\hat{q}_\epsilon[\psi] = \|\nabla\psi\|_{L^2(\omega)}^2 + (\psi, V\psi)_{L^2(\omega)},$$

where

$$V := \frac{\Delta\chi}{\chi} + \epsilon^2 \left(\frac{\partial_\tau \chi}{\chi} \right)^2.$$

Without the potential V , \hat{q}_ϵ would be just the form associated with the Dirichlet Laplacian $-\Delta_D^\omega$ in $L^2(\omega)$. The form domain of the latter is $H_0^1(\omega)$, which is compactly embedded in $L^2(\omega)$. It is thus enough to show that V is a relatively form bounded perturbation of $-\Delta_D^\omega$ due to the stability result [13, Thm. VI.3.4]. To do so, we use several facts:

- (i) By standard elliptic regularity theory (see, *e.g.*, [8, Sec. 6.3]), we have $\chi \in H^4(\omega)$. Consequently, $\nabla\chi \in H^3(\omega)$ and $\Delta\chi \in H^2(\omega)$. Using the Sobolev embedding [1, Thm. 5.4] $H^2(\omega) \hookrightarrow C^0(\bar{\omega})$, we thus have $\|\Delta\chi\|_\infty < \infty$ and $\|\partial_\tau \chi\|_\infty \leq a\|\nabla\chi\|_\infty < \infty$.
- (ii) For any domain ω such that $\partial\omega$ is of class C^2 , there exists [5, Lem. 4.6.1] a positive number α_0 such that $\chi \geq \alpha_0 d$, where $d(x') := \text{dist}(x', \partial\omega)$.
- (iii) For any strongly regular domain ω , which is in particular satisfied under the present smoothness assumption, the Hardy inequality $-\Delta_D^\omega \geq c_0/d^2$ holds true [5, Sec. 1.5].

Using (i)–(ii), we have

$$\begin{aligned} \left| \int_\omega \frac{\Delta\chi}{\chi} \psi^2 \right| &\leq \frac{\|\Delta\chi\|_\infty}{\alpha_0} \int_\omega \frac{\psi^2}{d} \leq \frac{\|\Delta\chi\|_\infty}{\alpha_0} \left(\delta \int_\omega \frac{\psi^2}{d^2} + \delta^{-1} \int_\omega \psi^2 \right), \\ \left| \int_\omega \epsilon^2 \left(\frac{\partial_\tau \chi}{\chi} \right)^2 \psi^2 \right| &\leq \epsilon^2 \frac{\|\partial_\tau \chi\|_\infty^2}{\alpha_0^2} \int_\omega \frac{\psi^2}{d^2}, \end{aligned}$$

for any $\psi \in C_0^\infty(\omega)$ and $\delta > 0$. Finally, using (iii), we deduce

$$\left| \int_\omega V \psi^2 \right| \leq b \|\nabla\psi\|_{L^2(\omega)}^2 + C \|\psi\|_{L^2(\omega)}^2,$$

where

$$b := \frac{1}{c_0} \left(\delta \frac{\|\Delta\chi\|_\infty}{\alpha_0} + \epsilon^2 \frac{\|\partial_\tau \chi\|_\infty^2}{\alpha_0^2} \right), \quad C := \delta^{-1} \frac{\|\Delta\chi\|_\infty}{\alpha_0}.$$

Hence, by taking δ and ϵ small enough, V is a relatively form bounded perturbation of $-\Delta_D^\omega$ with the relative bound b less than one. \square

Proposition 3. *Under the hypothesis and notation of Lemma 1, $\mu_\epsilon > 0$ for every $|\epsilon| \in (0, \epsilon_0)$, unless ω is rotationally symmetric.*

Proof. If $|\epsilon| < \epsilon_0$, the spectrum of h_ϵ is purely discrete. In particular, the spectral threshold μ_ϵ is an eigenvalue and the infimum (33) is attained by a corresponding eigenfunction $\phi \in L^2(\omega, \chi(x')^2 dx')$, i.e.,

$$\mu_\epsilon = \frac{\|\chi \nabla \phi\|_{L^2(\omega)}^2 + \epsilon^2 \|\partial_\tau \chi \phi\|_{L^2(\omega)}^2}{\|\chi \phi\|_{L^2(\omega)}^2}.$$

Assuming $\mu_\epsilon = 0$, it follows that $\|\chi \nabla \phi\|_{L^2(\omega)} = 0$ and $\|\partial_\tau \chi \phi\|_{L^2(\omega)} = 0$. From the first identity, since χ is positive, we deduce that ϕ is constant. Putting this result into the second identity, we conclude with $\|\partial_\tau \chi\|_{L^2(\omega)} = 0$, which is possible only if ω is rotationally symmetric with respect to the origin. \square

3.8 Uniform positivity in the cross-section

We come back to (30) and choose the parameters ε , β and a in such a way that c_1 and c_2 are positive. Note that the constants c_1 and c_2 can become only more positive if $\|\epsilon\|_\infty$ further diminishes. Employing the definition (33) and Fubini's theorem, we get

$$Q[\psi] \geq \frac{1}{2} \|\partial_1 \psi\|^2 + \|\sqrt{\mu} \psi\|^2, \quad (34)$$

where

$$\mu(x) := c_1 \mu_{\epsilon(x_1)} \quad \text{with} \quad \epsilon(x_1) := \sqrt{\frac{c_2}{c_1} \beta \varepsilon(x_1)}.$$

Let ω be different from a disc or annulus. Let $|\varepsilon|$ be non-trivial and so small on a bounded interval $I \subset \mathbb{R}$ such that $|\epsilon(x_1)| < \epsilon_0$ for almost every $x_1 \in I$. Then we know by Proposition 3 that $x_1 \mapsto \mu(x_1, x')$ is non-trivial and non-negative on I (by definition, $\mu(x)$ is independent of $x' \in \omega$). Consequently,

$$Q[\psi] \geq \frac{1}{4} \|\partial_1 \psi\|^2 + \nu \|\psi\|_{L^2(I \times \omega)}^2, \quad (35)$$

where ν is the lowest eigenvalue of the one-dimensional operator $-\frac{1}{4} \Delta_N^I + \mu$ in $L^2(I)$. Note that ν is positive because the potential μ is non-trivial and non-negative. Summing up, we have established the following crucial result.

Theorem 3. *Let $\beta \varepsilon \geq 0$ and $\beta \varepsilon \neq 0$. Suppose that ω is not rotationally invariant and that its boundary $\partial \omega$ is of class C^4 . In addition to (32), assume that there exists a bounded interval $I \subset \mathbb{R}$ such that*

$$0 < |\beta \epsilon(x_1)| < \frac{c_1}{c_2} \epsilon_0^2 \quad \text{for a.e. } x_1 \in I, \quad (36)$$

where c_1, c_2 are the constants depending on a , $|\beta|$ and $\|\varepsilon\|_\infty$ introduced in (31) and ϵ_0 is the number depending on the geometry of ω from Lemma 1. Then (35) holds for every $\psi \in H_0^1(\mathbb{R} \times \omega)$ with a positive number ν .

Notice that (35) is equivalent to the operator inequality

$$H - \lambda_1 \geq \left(-\frac{1}{4}\Delta^{\mathbb{R}} + \nu \chi_I\right) \otimes 1, \quad (37)$$

where χ_I is the characteristic function of I and the right hand side employs the Hilbert-space identification $L^2(\mathbb{R} \times \omega) \simeq L^2(\mathbb{R}) \otimes L^2(\omega)$. In particular, whenever ν is positive, we get another local Hardy-type inequality

$$H - \lambda_1 \geq \nu \chi_{I \times \omega}. \quad (38)$$

3.9 Global Hardy inequality

It is known how to deduce from (37) a global Hardy inequality with help of the classical result (1) (see [16] or [14, Sec. 7.2]). For the convenience of the reader and self-consistency, we repeat the procedure here.

Theorem 4. *Under the hypotheses of Theorem 3, there exists a positive constant c depending on β , the geometry of ω and properties of ε such that*

$$H - \lambda_1 \geq \frac{c}{1 + x_1^2} \quad (39)$$

holds in the form sense in $L^2(\mathbb{R} \times \omega)$.

Proof. Let x_1^0 denote the centre of the interval I . The main ingredient in the proof is the following Hardy-type inequality for a Schrödinger operator in $\mathbb{R} \times \omega$ with a characteristic-function potential:

$$\|\rho\psi\|^2 \leq 16 \|\partial_1\psi\|^2 + (2 + 64/|I|^2) \|\psi\|_{L^2(I \times \omega)}^2 \quad (40)$$

for every $\psi \in H_0^1(\mathbb{R} \times \omega)$, where we denote $\rho(x) := 1/\sqrt{1 + (x_1 - x_1^0)^2}$. This inequality is a consequence of (1). Indeed, following [7, Sec. 3.3], let η be the Lipschitz function on \mathbb{R} defined by $\eta(x_1) := 2|x_1 - x_1^0|/|I|$ for $|x_1 - x_1^0| \leq |I|/2$ and 1 otherwise. For any $\psi \in C_0^\infty(\mathbb{R} \times \omega)$, let us write $\psi = \eta\psi + (1 - \eta)\psi$, so that $(\eta\psi)(\cdot, x') \in H_0^1(\mathbb{R} \setminus \{x_1^0\})$ for every $x' \in \omega$. Then, employing Fubini's theorem, we can estimate as follows:

$$\begin{aligned} \|\rho\psi\|^2 &\leq 2 \int_{\mathbb{R} \times \omega} (x_1 - x_1^0)^{-2} |(\eta\psi)(x)|^2 dx + 2 \|(1 - \eta)\psi\|^2 \\ &\leq 8 \|\partial_1(\eta\psi)\|^2 + 2 \|\psi\|_{L^2(I \times \omega)}^2 \\ &\leq 16 \|\eta\partial_1\psi\|^2 + 16 \|(\partial_1\eta)\psi\|^2 + 2 \|\psi\|_{L^2(I \times \omega)}^2 \\ &\leq 16 \|\partial_1\psi\|^2 + (2 + 64/|I|^2) \|\psi\|_{L^2(I \times \omega)}^2. \end{aligned}$$

By density, this result extends to all $\psi \in H_0^1(\mathbb{R} \times \omega) = \mathfrak{D}(Q) = \mathfrak{D}(h)$.

By Theorem 3, we have

$$Q[\psi] \geq \frac{1 - \delta}{4} \|\partial_1\psi\|^2 + \frac{\delta}{4} \|\partial_1\psi\|^2 + \nu \|\psi\|_{L^2(I \times \omega)}^2$$

for every $\psi \in H_0^1(\mathbb{R} \times \omega)$ and $\delta \in (0, 1]$, where ν is a positive number. Neglecting the first term on the right hand side and using (40), we get

$$Q[\psi] \geq \frac{\delta}{64} \|\rho\psi\|^2 + \left(\nu - \frac{\delta}{32}(1 + 32/|I|^2)\right) \|\psi\|_{L^2(I \times \omega)}^2.$$

Employing the positivity of ν , we choose $\delta = \min\{1, 32\nu/(1+32/|I|^2)\}$ and thus obtain

$$H - \lambda_1 \geq \frac{c'}{1 + (x_1 - x_1^0)^2}$$

with $c' = \delta/64$.

To conclude with (39), we set

$$c := c' \min_{x_1 \in \mathbb{R}} \frac{1 + x_1^2}{1 + (x_1 - x_1^0)^2},$$

where the minimum is a positive constant depending on x_1^0 . \square

In view of the unitary equivalence between H and $-\Delta_D^\Omega$ and since the longitudinal coordinate x_1 is left invariant by the rotation matrix in (2), (39) is equivalent to the operator inequality

$$-\Delta_D^\Omega - \lambda_1 \geq \frac{c}{1 + x_1^2}$$

in the form sense in $L^2(\Omega)$, with the same constant c . This establishes Theorem 1 as a consequence of Theorem 4 by noticing that $|x_1| \leq |x|$, where, with an abuse of notation, $|x|$ stand for the magnitude of the radial vector in $\mathbb{R}^3 \supset \Omega$.

A Absence of eigenvalues in a one-dimensional problem

In this appendix we give a proof of Proposition 1. Redefining β and ε in (12), we may assume, without loss of generality, that the constant C_ω is equal to 1 and consider in $L^2(\mathbb{R})$ just the Schrödinger operator $H_\alpha := -\Delta^\mathbb{R} + V_\alpha$, $\mathfrak{D}(H_\alpha) = \mathfrak{D}(-\Delta^\mathbb{R}) = H^2(\mathbb{R})$, with the potential

$$V_\alpha := (\alpha\varepsilon + \beta)^2 - \beta^2 = \alpha^2\varepsilon^2 + 2\alpha\beta\varepsilon.$$

We make the hypothesis that $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function of compact support and $\beta \in \mathbb{R}$. Since the sign of β is not restricted, we may assume, again without loss of generality, that the coupling parameter α is positive. We prove Proposition 1 by considering the operator H_α in the limit as $\alpha \rightarrow \infty$.

The support of ε is a closure of a countable union of bounded open intervals I_n . Let $H_\alpha^{I_n}$ be the operator in $L^2(I_n)$ that acts as H_α inside I_n and satisfies the Neumann boundary conditions at ∂I_n . That is, $H_\alpha^{I_n} = -\Delta_N^{I_n} + V_\alpha$, where $-\Delta_N^{I_n}$ is the Neumann Laplacian in $L^2(I_n)$. Clearly,

$$\inf \sigma(H_\alpha) \geq \min \left\{ 0, \inf_n \inf \sigma(H_\alpha^{I_n}) \right\}. \quad (41)$$

Here $\inf \sigma(H_\alpha^{I_n})$ is just the lowest eigenvalue of $H_\alpha^{I_n}$, since the latter is an operator with compact resolvent. Our strategy to prove Proposition 1 is to show that each $\inf \sigma(H_\alpha^{I_n})$ is positive for all sufficiently large α .

We shall need the following auxiliary result.

Lemma 2. Let $-\infty < a < a' < b' < b < +\infty$. Then

$$[-\Delta_N^{(a,b)} + \alpha\chi_{(a',b')}]^{-1} \xrightarrow[\alpha \rightarrow \infty]{s} [-\Delta_{ND}^{(a,a')}]^{-1} \oplus 0 \oplus [-\Delta_{DN}^{(b',b)}]^{-1},$$

where D, N stand for the Dirichlet and Neumann boundary conditions at the respective parts of the interval and the direct-sums are with respect to the decomposition $L^2((a, b)) \simeq L^2((a, a')) \oplus L^2((a', b')) \oplus L^2((b', b))$.

Proof. Given $F \in L^2((a, b))$, let $\psi_\alpha := [-\Delta_N^{(a,b)} + \alpha\chi_{(a',b')} + 1]^{-1}F$. It satisfies the weak formulation of the resolvent equation

$$\int_a^b \bar{v}'(x) \psi'_\alpha(x) dx + \alpha \int_{a'}^{b'} \bar{v}(x) \psi_\alpha(x) dx + \int_a^b \bar{v}(x) \psi_\alpha(x) dx = \int_a^b \bar{v}(x) F(x) dx \quad (42)$$

for every $v \in H^1((a, b))$. Choosing $v = \psi_\alpha$, we get

$$\int_a^b |\psi'_\alpha(x)|^2 dx + \alpha \int_{a'}^{b'} |\psi_\alpha(x)|^2 dx + \int_a^b |\psi_\alpha(x)|^2 dx = \int_a^b \bar{\psi}_\alpha(x) F(x) dx,$$

from which we deduce

$$\|\psi_\alpha\|_{H^1((a,b))}^2 + \alpha \|\psi_\alpha\|_{L^2((a',b'))}^2 \leq \|\psi_\alpha\|_{L^2((a,b))} \|F\|_{L^2((a,b))}.$$

It follows that $\{\psi_\alpha\}_{\alpha>0}$ is a bounded family in $H^1((a, b))$ and therefore pre-compact in the weak topology of this space. Let ψ_∞ be a weak limit point as $\alpha \rightarrow \infty$. That is, for an increasing sequence of positive numbers $\{\alpha_k\}_{k \in \mathbb{N}}$ such that $\alpha_k \rightarrow \infty$ as $k \rightarrow \infty$, $\{\psi_{\alpha_k}\}_{k \in \mathbb{N}}$ converges weakly in $H^1((a, b))$ to ψ_∞ . In fact, we may assume that it converges strongly in $L^2((a, b))$, because the embedding $H^1((a, b)) \hookrightarrow L^2((a, b))$ is compact. Dividing by α_k , we also see that

$$\|\psi_{\alpha_k}\|_{L^2((a',b'))}^2 \longrightarrow 0$$

as $k \rightarrow \infty$. Consequently, $\psi_\infty = 0$ on $[a', b']$ (recall the embedding $H^1((a, b)) \hookrightarrow C^0([a, b])$). In particular, ψ_∞ satisfies the Dirichlet boundary conditions at a', b' . Choosing in (42) a test function v that vanish on $[a', b']$, restricting to the subsequence ψ_{α_k} and taking the limit $k \rightarrow \infty$, we get that ψ_∞ is a solution to the boundary-value problem

$$\begin{cases} -\psi''_\infty + \psi_\infty = F & \text{in } (a, a') \cup (b', b), \\ \psi'_\infty = 0 & \text{at } a, b, \\ \psi_\infty = 0 & \text{at } a', b'. \end{cases}$$

We have thus verified $\psi_\infty = [-\Delta_{ND}^{(a,a')}]^{-1}F \oplus 0 \oplus [-\Delta_{DN}^{(b',b)}]^{-1}F$. The same limit is obtained for *any* weak limit point of $\{\psi_\alpha\}_{\alpha>0}$. Consequently, ψ_α converges strongly in $L^2((a, b))$ to ψ_∞ as $\alpha \rightarrow \infty$, which was to be proved. \square

Since the resolvents of Lemma 2 are compact, we get convergence of eigenvalues, in particular:

Corollary 1.

$$\inf \sigma(-\Delta_N^{(a,b)} + \alpha\chi_{(a',b')}) \xrightarrow[\alpha \rightarrow +\infty]{} \min \left\{ \left(\frac{\pi}{2(a' - a)} \right)^2, \left(\frac{\pi}{2(b - b')} \right)^2 \right\}.$$

Now, recalling (41), fix n and consider the lowest eigenvalue $\inf \sigma(H_\alpha^{I_n})$ of $H_\alpha^{I_n}$. Let I'_n be any open subinterval of I_n such that $\overline{I'_n} \subset I_n$. Then there exist positive constants $c_n = c_n(\beta, \varepsilon \upharpoonright I'_n)$ and $\alpha_n = \alpha_n(\beta, \varepsilon \upharpoonright I'_n)$ such that $V_\alpha \geq c_n \alpha^2$ on I'_n for all $\alpha \geq \alpha_n$. At the same time, $V_\alpha \geq -\beta^2$ on the whole interval I_n . Consequently,

$$H_\alpha^{I_n} \geq -\Delta_N^{I_n} + c_n \alpha^2 \chi_{I'_n} - \beta^2 \chi_{I_n \setminus I'_n},$$

where on the right hand side there is an operator to which Lemma 2 and its Corollary 1 apply. It follows that, choosing I'_n in such a way that $|I_n \setminus I'_n|$ is sufficiently small, we get that $\inf \sigma(-\Delta_N^{I_n} + c_n \alpha^2 \chi_{I'_n})$ will be larger than β^2 for all sufficiently large α , and thus $\inf \sigma(H_\alpha^{I_n})$ positive.

In general, the problem is that the constants measuring the largeness of α depend on n , so that we have no uniform control over the infimum appearing in (41). This problem of course does not appear under the hypothesis of Proposition 1 that the support of ε is just a closure of a *finite* union of open intervals (recall that in general it is just a *countable* union). Therefore we get the desired result under this extra assumption.

B The Neumann bracketing lowers the spectrum too much

For the straight tube (*i.e.* $\beta = 0 = \varepsilon$), imposing an extra Neumann condition at $\{x_1 = 0\}$ does not change the spectrum of the Dirichlet Laplacian in Ω . This fact is used in [17, 20] to go from the positivity of $H - E_1$ for a locally twisted tube (*i.e.* $\beta = 0$) to the Hardy inequality (3). In this appendix we demonstrate that this trick does not seem to be useful for periodically twisted tubes (*i.e.* $\beta \neq 0$) investigated in this paper. Indeed, by imposing the supplementary Neumann condition at $\{x_1 = 0\}$, one creates a spectrum below λ_1 in (6).

Imposing the “Neumann condition” at $\{x_1 = 0\}$ of the tube (2) means that one considers instead of H the operator H^N which is associated in $L^2(\mathbb{R} \times \omega)$ with the form h^N that acts as h in (14) but has a larger domain

$$\mathfrak{D}(h^N) := \mathfrak{D}_- \oplus \mathfrak{D}_+,$$

where \mathfrak{D}_+ (respectively, \mathfrak{D}_-) denotes the set of functions from $H_0^1(\mathbb{R} \times \omega)$ restricted to $(0, \infty) \times \omega$ (respectively, $(-\infty, 0) \times \omega$). Obviously, $H \geq H^N$. However, this comparison is not much useful in view of the following result.

Proposition 4. *Let $\beta \neq 0$ and $\varepsilon = 0$. Then*

$$\inf \sigma(H^N) < \lambda_1.$$

Proof. The proof is based on the variational idea to construct a test function $\psi \in \mathfrak{D}(h^N)$ such that

$$Q^N[\psi] := h^N[\psi] - \lambda_1 \|\psi\|^2 < 0.$$

First, we check that Q^N can be made asymptotically vanishing for a suitably chosen sequence of test functions. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such

that $\varphi(x_1) = 1$ for $x_1 \in [-1, 1]$ and $\varphi(x_1) = 0$ for $|x_1| \geq 2$. Given any natural number $n \geq 1$, set $\varphi_n(x_1) := \varphi(x_1/n)$. Note that $\varphi_n \chi \in \mathfrak{D}(h) \subset \mathfrak{D}(h^N)$. Then

$$Q^N[\varphi_n \chi] = \|\dot{\varphi}_n \chi\|^2 - 2(\dot{\varphi}_n \chi, \beta \varphi_n \partial_\tau \chi) = \|\dot{\varphi}_n\|_{L^2(\mathbb{R})}^2 = \frac{1}{n} \|\dot{\varphi}\|_{L^2(\mathbb{R})}^2 \xrightarrow{n \rightarrow \infty} 0.$$

Here the second equality follows by an integration by parts and the normalisation of χ . Next, we add a small perturbation

$$\psi(x) := \varphi_n(x_1) \chi(x') + \delta \phi(x),$$

where δ is a small real parameter and $\phi \in \mathfrak{D}(h^N)$ will be specified later. Choosing ϕ real-valued, we can write

$$Q^N[\psi] = Q^N[\varphi_n \chi] + 2\delta Q^N(\phi, \varphi_n \chi) + \delta^2 Q^N[\phi]. \quad (43)$$

Now we specify

$$\phi(x) := \rho(x_1) \tau(x') \chi(x'), \quad \text{where} \quad \tau(x') := -\arctan \frac{x_3}{x_2}$$

is the angular variable and ρ is a real-valued function supported in $[-1, 0]$ such that $\rho(0) \neq 0$. Note that ϕ belongs to $\mathfrak{D}(h^N)$, although it does not belong to $\mathfrak{D}(h)$. Integrating by parts in transverse variables and employing the eigenvalue equation χ satisfies, it is easy to check

$$Q^N(\phi, \varphi_n \chi) = (\partial_1 \phi, \dot{\varphi}_n \chi) - (\partial_1 \phi, \beta \varphi_n \partial_\tau \chi) + (\beta \phi, \dot{\varphi}_n \partial_\tau \chi).$$

Since ρ is supported in the interval where $\varphi_n = 1$, the first and last integrals on the right hand side equal zero. Integrating by parts in the remaining integral and using the normalisation of χ , we conclude with

$$Q^N(\phi, \varphi_n \chi) = -(\partial_1 \phi, \beta \varphi_n \partial_\tau \chi) = -\beta \rho(0) \int_\omega \tau \chi \partial_\tau \chi = \frac{\beta \rho(0)}{2}.$$

To fix the sign of the result, let us choose $\rho(0) = -\beta$. Then the mixed term on the right hand side of (43) is negative and independent of n for all positive δ . Moreover, the sum with the last term can be guaranteed to remain negative by choosing δ sufficiently small. Finally, we choose n sufficiently large in order to make the sum of all terms in (43) negative. \square

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